1. HF / Analízis / ÁVF / Lev. / 2006. őszi félév

Megoldások

I. Számsorozatok

1. $a_1=-\frac{3}{2},\ a_2=1,\ a_3=-\frac{3}{4},\ a_4=\frac{3}{5},\ a_5=-\frac{1}{2}$ $\quad\left(a_n=(-1)^n\frac{3}{n+1}\right)$

$b_1=\frac{5}{2},\ b_2=\frac{10}{9},\ b_3=\frac{15}{28},\ b_4=\frac{20}{65},\ b_5=\frac{25}{126}$ $\quad\left(b_n=\frac{5n}{n^3+1}\right)$

$c_1=\frac{1}{3},\ c_2=\frac{1}{9},\ c_3=\frac{1}{27},\ c_4=\frac{1}{81},\ c_5=\frac{1}{243}$ $\quad\left(c_n=\left(\frac{1}{3}\right)^n\right)$

$d_1=3,\ d_2=9,\ d_3=27,\ d_4=81,\ d_5=243$ $\quad\left(d_n=3^n\right)$

$f_0=2,\ f_1=\sqrt{8},\ f_2=\sqrt{\sqrt{8}+6}\approx 2{,}97,\ f_3=\sqrt{8{,}97}\approx 2{,}995$

$f_4\approx 2{,}999,\ f_5\approx 2{,}9998$

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2. $a_n=?$

a) $a_n=-2n$ b) $a_n=\left(-\frac{1}{2}\right)^{n-1}$, c) $a_n=\frac{5}{n}$

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3. $a_n=(-1)^n\frac{3}{n+1}$ korlátos, nem monoton sorozat : $|a_n|<2$

$b_n=\frac{5n}{n^3+1}$, $0<b_n\le\frac{5}{2}$, korlátos és szig. mon. fogyó, u.is

$$\left(b_n-b_{n+1}=\frac{5n}{n^3+1}-\frac{5(n+1)}{(n+1)^3+1}=\frac{5\left[n(n+1)(2n+1)-1\right]}{(n^3+1)\cdot\left[(n+1)^3+1\right]}>0\right)$$

$c_n=\left(\frac{1}{3}\right)^n$ szig. mon. fogyó és korlátos: $0<\left(\frac{1}{3}\right)^n\le\frac{1}{3}$

$d_n=3^n$ szig. mon. növő, felülről nem korlátos, $0\le d_n$

$f_0=2,\ f_{n+1}=\sqrt{f_n+6}$ szig. mon. növő, korlátos sorozat : $0<f_n<3$

Biz. teljes indukcióval

szig. mon. nő : $f_1=\sqrt{8}\approx 2{,}828<f_2\approx 2{,}97$

$$f_{n+1}=\sqrt{f_n+6}\underset{\text{ind. felt.}}{>}\sqrt{f_{n-1}+6}=f_n$$

Sorozatos: $f_1=\sqrt{8}\approx 2{,}828<3$

tfh. $f_n<3$

$$f_{n+1}=\sqrt{f_n+6}\overset{\text{ind. felt}}{<}\sqrt{3+6}=\sqrt{9}=3.$$

4\. a) $$\lim\frac{-n^3+3n^2-100}{(n+1)^3+2n+1}=\lim\frac{-1+\frac{3}{n}-\frac{100}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{2}{n^2}+\frac{1}{n^3}}=-1$$

b) $$\lim\frac{\sqrt{n^2+2}+n\sqrt[3]{n}}{2n^{\frac{4}{3}}+5\sqrt{n}}=\lim\frac{\frac{\sqrt{n^2+2}}{n\sqrt[3]{n}}+1}{2+5\frac{\sqrt{n}}{n\sqrt[3]{n}}}=\frac{1}{2}$$

$$\left(\frac{\sqrt{n^2+2}}{n\sqrt[3]{n}}=\frac{\sqrt{1+\frac{2}{n^2}}}{\sqrt[3]{n}}\to\frac{1}{\infty}=0\ ,\quad \frac{\sqrt{n}}{n\sqrt[3]{n}}=\frac{1}{\sqrt{n}\cdot\sqrt[3]{n}}\to\frac{1}{\infty}=0\right)$$

c) $$\lim\frac{(\sqrt{2n+1}-\sqrt{n})(\sqrt{2n+1}+\sqrt{n})(\sqrt{n-2}+\sqrt{n})}{(\sqrt{n-2}-\sqrt{n})(\sqrt{n-2}+\sqrt{n})(\sqrt{2n+1}+\sqrt{n})}=\lim\frac{(2n+1-n)(\sqrt{n+2}+\sqrt{n})}{(n+2-n)(\sqrt{2n+1}+\sqrt{n})}=$$

$$=\lim\frac{n+1}{2}\cdot\frac{\left(\sqrt{1+\frac{2}{n}}+1\right)}{\left(\sqrt{2+\frac{1}{n}}+1\right)}=+\infty\cdot\frac{2}{(\sqrt{2}+1)}=+\infty$$

d) $$\lim\sqrt{n+2}\,\frac{\left(\sqrt{\frac{n}{3}+1}-\sqrt{\frac{n}{3}-1}\right)\cdot\left(\sqrt{\frac{n}{3}+1}+\sqrt{\frac{n}{3}-1}\right)}{\sqrt{\frac{n}{3}+1}+\sqrt{\frac{n}{3}-1}}=\lim\frac{2\sqrt{n+2}}{\sqrt{\frac{n}{3}+1}+\sqrt{\frac{n}{3}-1}}$$

$$=\lim\frac{2\sqrt{1+\frac{2}{n}}}{\sqrt{\frac{1}{3}+\frac{1}{n}}+\sqrt{\frac{1}{3}-\frac{1}{n}}}=\frac{2}{2\sqrt{\frac{1}{3}}}=\sqrt{3}$$

e) $$\lim\frac{3\cdot3^n-2\cdot2^n}{2\cdot3^n+\frac{1}{9}\cdot9^n+2\cdot4^n}=\lim\frac{3\left(\frac{1}{3}\right)^n-2\cdot\left(\frac{2}{9}\right)^n}{2\left(\frac{1}{3}\right)^n+\frac{1}{9}+2\cdot\left(\frac{4}{9}\right)^n}=0=\frac{0}{\frac{1}{9}}$$

f) $$\lim\left(\frac{(n+3)-1}{n+3}\right)^{n-1}=\lim\left[\left(1+\frac{-1}{n+3}\right)^{n+3}\right]\cdot\left(1+\frac{-1}{n+3}\right)^{-4}=e^{-1}\cdot1=\frac{1}{e}$$

II. Végtelen sorok

1. a) $\sum_{n=1}^{\infty} \frac{9\cdot 3^n}{\frac{1}{36}\cdot 6^n} = 9\cdot 36 \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{324}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 162\cdot\frac{1}{1-\frac{1}{2}} = 324$

b) $\sum_{n=2}^{\infty} \frac{2\cdot 2^n\cdot \frac{1}{3}\cdot 3^n}{2\cdot 6^n\cdot 4^n} = \frac{1}{3}\sum_{n=2}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{48}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{48}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{36}$

c) $\sum_{n=1}^{\infty} 8\cdot 2^n\cdot 16\cdot\frac{1}{16^n} = 128\sum_{n=1}^{\infty}\left(\frac{1}{8}\right)^n = 16\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = 16\cdot\frac{1}{1-\frac{1}{8}} = \frac{128}{7}$

d) $\sum_{k=0}^{\infty} 2^k \overset{D}{=} \lim \sum_{k=0}^{n} 2^k = \lim\frac{2^{n+1}-1}{2-1} = \lim\left(2^{n+1}-1\right) = +\infty$

e) $\sum_{k=0}^{\infty} (-1)^k\cdot 3 = 3-3+3-3+3\mp$ nem létezik, u.i. a részlet-összegek sorozata a köv.: $s_1 = 3-3 = 0$

$s_2 = 3-3+3 = 3$

$s_3 = 3-3+3-3 = 0$

stb.

azaz $s_{2k} = 3$ és $s_{2k+1} = 0$ $\forall k$-ra, ennek a sorozatnak pedig nincs limesze!

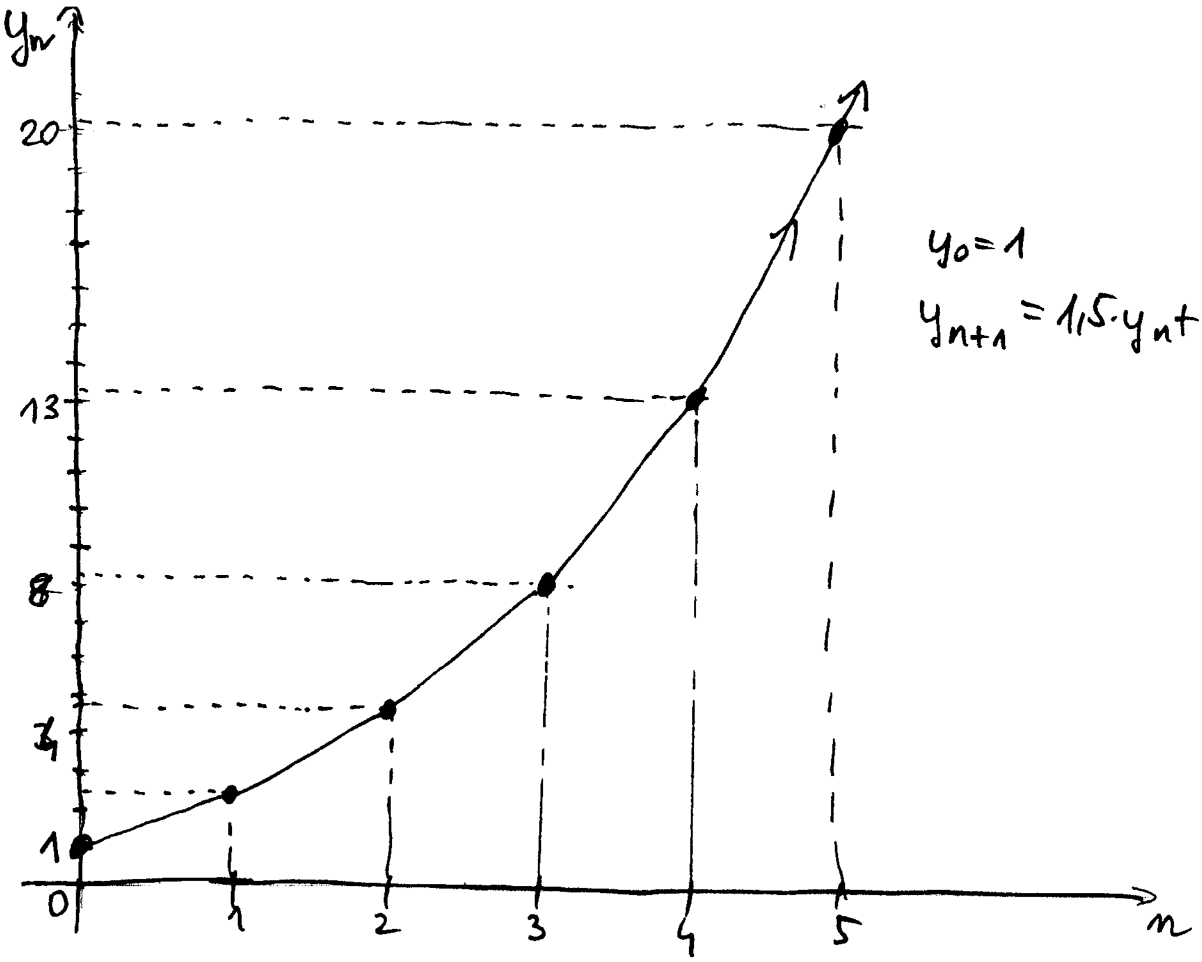
III. DaE

1. $y_0 = 1$, $y_{n+1} = 1{,}5\cdot y_n + 1$

$y_1 = 2{,}5$, $y_2 = 4{,}75$, $y_3 = 8{,}125$, $y_4 = 13{,}1875$, $y_5 = 20{,}78125$

$$y_n = 1{,}5^n\left(1-\frac{1}{1-1{,}5}\right) + \frac{1}{1-1{,}5} = 3\cdot 1{,}5^n - 2$$

$$\lim y_n = \lim\left(3\cdot 1{,}5^n - 2\right) = +\infty.$$



$y_0 = 1$

$y_{n+1} = 1{,}5 \cdot y_n + 1$

b) $y_0 = 4$

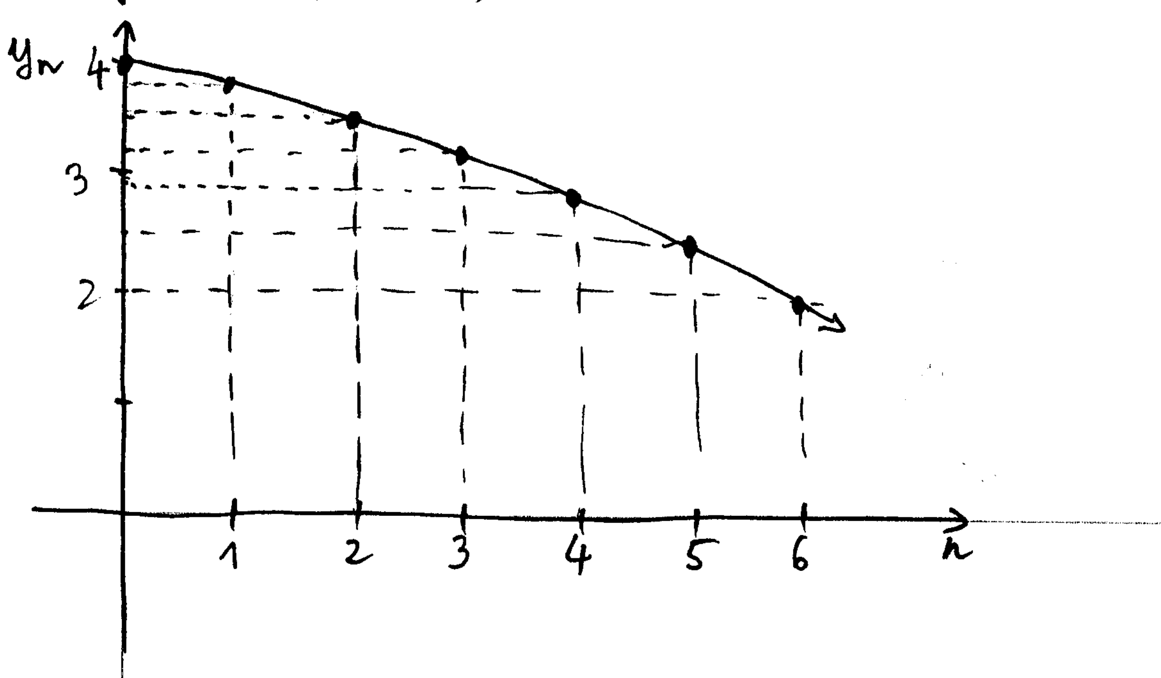
$y_{n+1} = 1{,}2 \cdot y_n - 1$

$y_1 = 3{,}8$ $\quad$ $y_3 = 3{,}272$ $\quad$ $y_5 = 2{,}51168$

$y_2 = 3{,}56$ $\quad$ $y_4 = 2{,}9264$ $\quad$ $y_6 = 2{,}014$

$$y_n = 1{,}2^n\left(4 - \frac{-1}{1-1{,}2}\right) + \frac{-1}{1-1{,}2} = -1{,}2^n + 5$$

$$\lim y_n = \lim\left(5 - 1{,}2^n\right) = -\infty$$

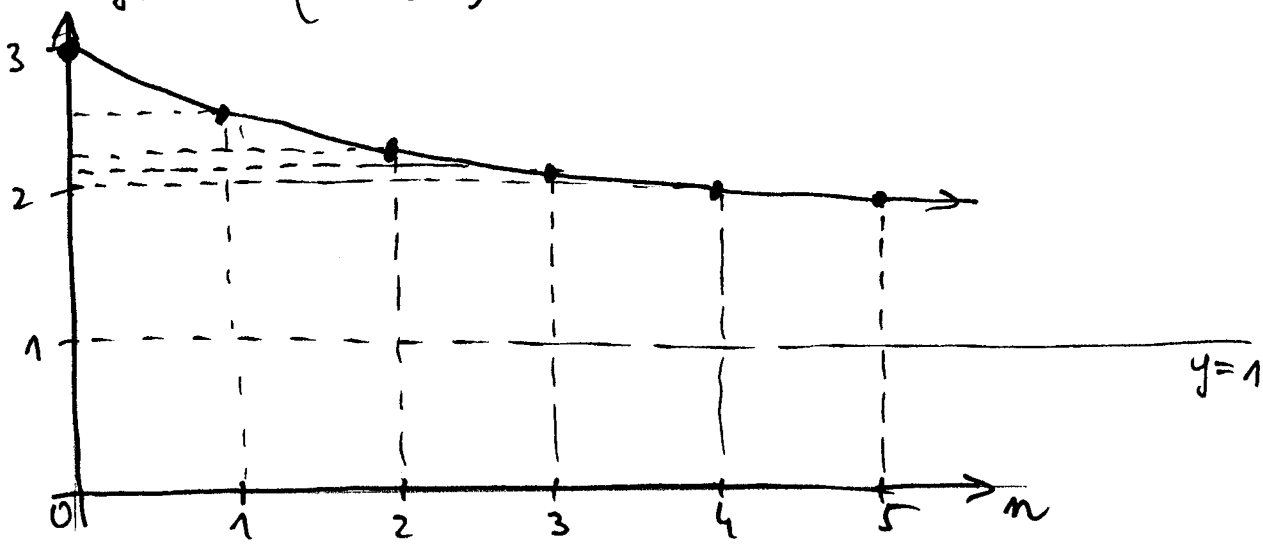


c) $y_0 = 3$
$y_{n+1} = 0{,}5 \cdot y_n + 1$

$y_1 = 2{,}5$	$y_3 = 2{,}125$	$y_5 = 2{,}03125$
$y_2 = 2{,}25$	$y_4 = 2{,}0625$	$y_6 = 2{,}015$

$$y_n = \left(\frac{1}{2}\right)^n \cdot \left(3 - \frac{1}{1-0{,}5}\right) + \frac{1}{1-0{,}5} = 1 + \left(\frac{1}{2}\right)^n$$

$$\lim y_n = \lim \left(1 + \left(\frac{1}{2}\right)^n\right) = 1$$

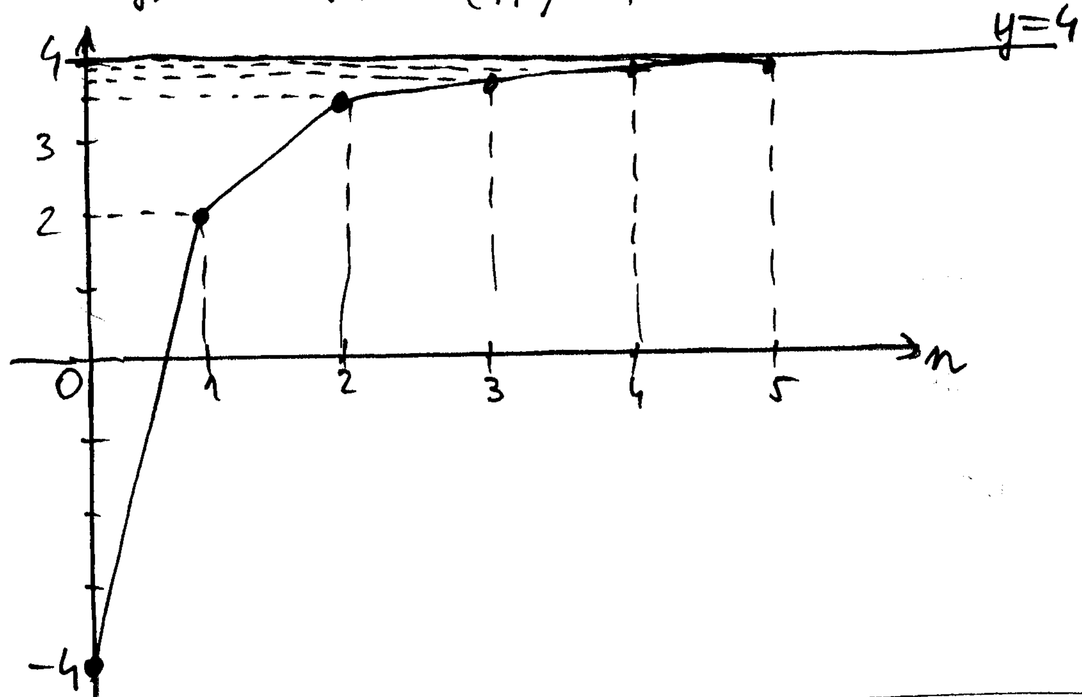


d) $y_0 = -4$
$y_{n+1} = 0{,}25 \cdot y_n + 3$

$y_1 = 2$	$y_3 = 3{,}875$	$y_5 = 3{,}992$
$y_2 = 3{,}5$	$y_4 = 3{,}96875$	

$$y_n = \left(\frac{1}{4}\right)^n \left(-4 - \frac{3}{1-\frac{1}{4}}\right) + \frac{3}{1-\frac{1}{4}} = 4 - 8 \cdot \left(\frac{1}{4}\right)^n$$

$$\lim y_n = \lim \left(4 - 8 \cdot \left(\frac{1}{4}\right)^n\right) = 4$$



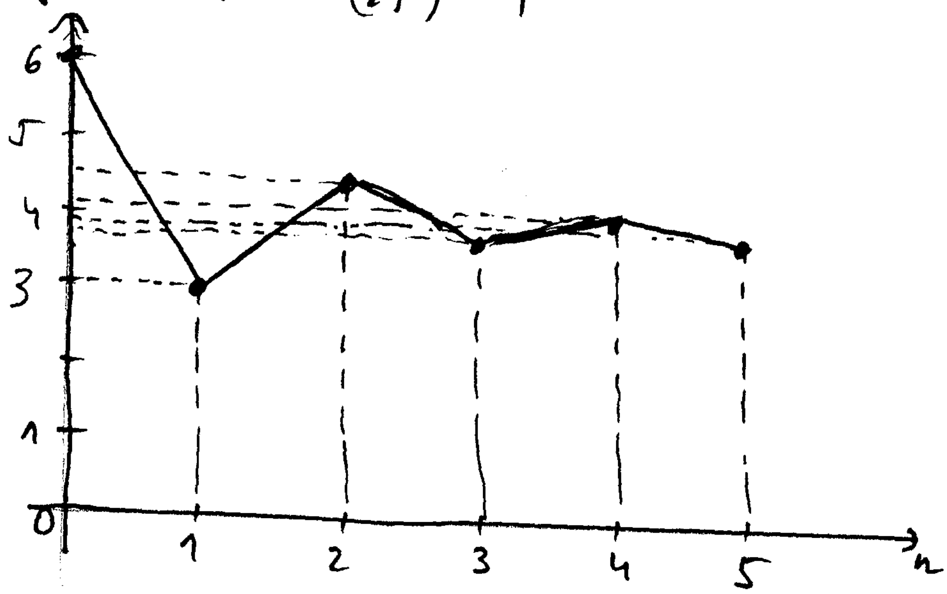
e) $y_0=6$

$y_{n+1}=-\frac{1}{2}y_n+6$

$y_1=3$ $\quad y_2=4{,}5$ $\quad y_3=3{,}75$ $\quad y_4=4{,}125$ $\quad y_5=3{,}9375$

$$y_n=\left(-\frac{1}{2}\right)^n\left(6-\frac{6}{1+0{,}5}\right)+\frac{6}{1+0{,}5}=4+2\cdot\left(\frac{1}{2}\right)^n$$

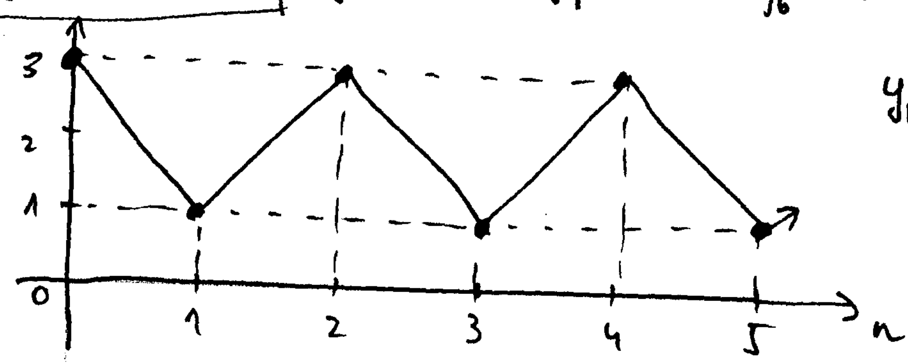
$$\lim y_n=\lim\left(4+2\cdot\left(\frac{1}{2}\right)^n\right)=4$$



f) $y_0=3$

$y_{n+1}=-y_n+4$

$y_1=1$ $\quad y_2=3$ $\quad y_3=1$ $\quad y_4=3$ $\quad y_5=1$ $\quad y_6=3$



$$y_n=(-1)^n\left(3-\frac{4}{1+1}\right)+2=$$
$$=2+(-1)^n$$

$\lim y_n$ nem létezik

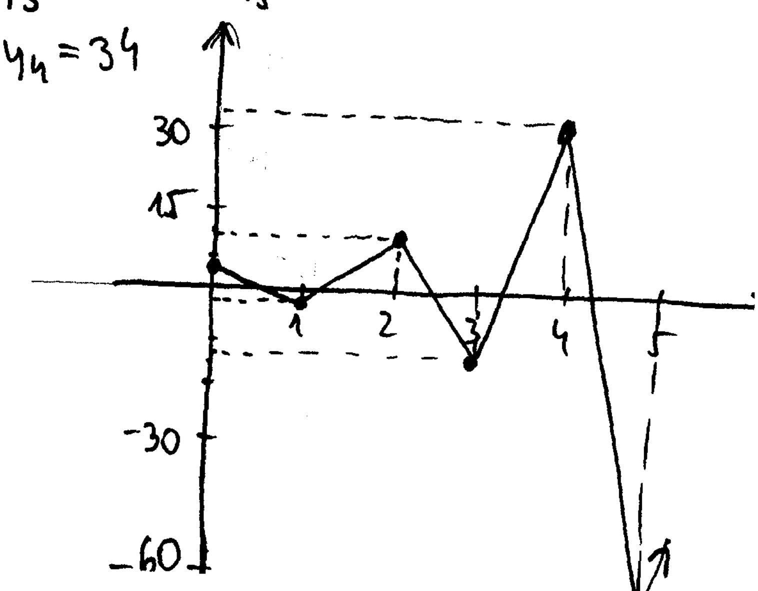
g) $y_0=4$

$y_{n+1}=-2y_n+6$

$y_1=-2$ $\quad y_2=10$ $\quad y_3=-14$ $\quad y_4=34$ $\quad y_5=-62$

$$y_n=(-2)^n\left(4-\frac{6}{1+2}\right)+2=2+2\cdot(-2)^n$$

$\lim y_n$ nem létezik!

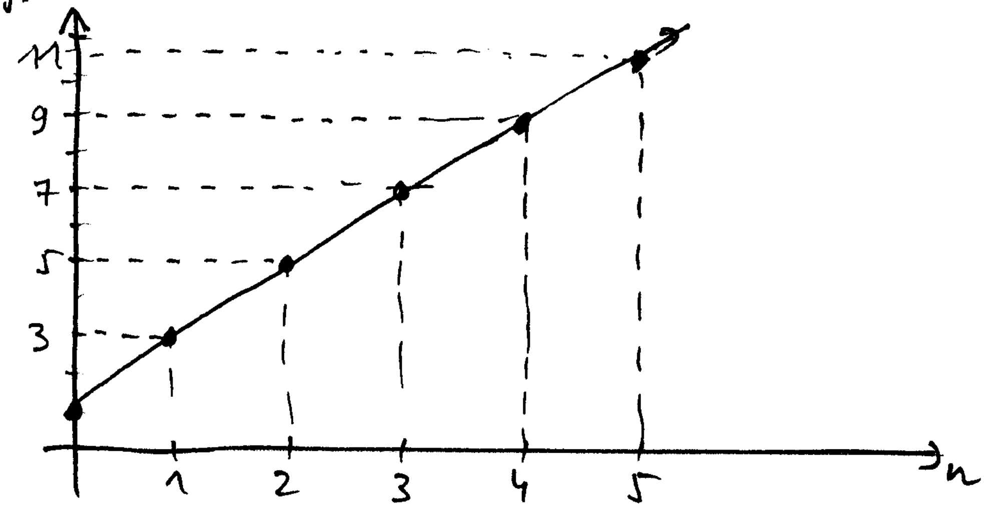


b) $y_0 = 1$ | $y_1 = 3$ $\quad y_3 = 7$ $\quad y_5 = 11$

$y_{n+1} = y_n + 2$ | $y_2 = 5$ $\quad y_4 = 9$

$y_n = 1 + n \cdot 2 = 1 + 2n$ (számtani sorozat)

$\lim y_n = \lim (1 + 2n) = +\infty$



② a) $y_{n+2} + y_{n+1} - 2y_n = 0$ | $\lambda^2 + \lambda - 2 = 0$ karakt. egy.

$y_0 = 1,\ y_1 = 2$ | $(\lambda + 2)(\lambda - 1) = 0$

$\lambda_1 = 1,\ \lambda_2 = -2$

$y_n = c_1 + c_2 (-2)^n$

$n=0: \quad 1 = c_1 + c_2$ 1)

$n=1 \quad 2 = c_1 - 2c_2$ 2)

2) − 1) : $1 = -3c_2$, $c_2 = -\frac{1}{3}$

$c_1 = \frac{4}{3}$

Tehát $y_n = \frac{4}{3} - \frac{1}{3} \cdot (-2)^n$

b) $y_{n+2} + 2y_{n+1} - 3y_n = 0$ | $\lambda^2 + 2\lambda - 3 = 0$ karakt. egyenlet

$(\lambda + 3)(\lambda - 1) = 0$, $\lambda_1 = 1,\ \lambda_2 = -3$

$y_n = c_1 + c_2 (-3)^n$

c) $y_{n+2} - y_{n+1} - 12y_n = 30$ | $\lambda^2 - \lambda - 12 = 0 = (\lambda - 4)(\lambda + 3)$

$y_0 = -1,\ y_1 = 3$ | $\lambda_1 = 4,\ \lambda_2 = -3$

$y_n = c_1 \cdot 4^n + c_2 (-3)^n + \frac{30}{1 - 1 - 12} = c_1 \cdot 4^n + c_2 \cdot (-3)^n - 2{,}5$

$n=0: \quad -1 = c_1 + c_2 - 2{,}5$ $\quad c_1 = 1{,}5 - c_2$, $c_2 = \frac{1}{14}$, $c_1 = \frac{20}{14}$

$n=1: \quad 3 = 4c_1 - 3c_2 - 2{,}5$

$$y_n = \frac{10}{7}\cdot 4^n + \frac{1}{14}\cdot(-3)^n - 2{,}5$$

IV. 1) $\lim\limits_{1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim\limits_{1}(x^2+x+1) = 3$

$\lim\limits_{-3+} \frac{|x+3|}{x+3} = 1 \qquad \left(\frac{|x+3|}{x+3} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases} \right) \qquad \lim\limits_{-3-} \frac{|x+3|}{x+3} = -1$

$\lim\limits_{2+} \frac{1}{2-x} = \frac{1}{0-} = -\infty, \quad \lim\limits_{2+} 2^{\frac{1}{2-x}} = \lim\limits_{y\to-\infty} 2^y = 0$

$\lim\limits_{2-} \frac{1}{2-x} = \frac{1}{0+} = +\infty, \quad \lim\limits_{2-} 2^{\frac{1}{2-x}} = \lim\limits_{y\to+\infty} 2^y = +\infty$

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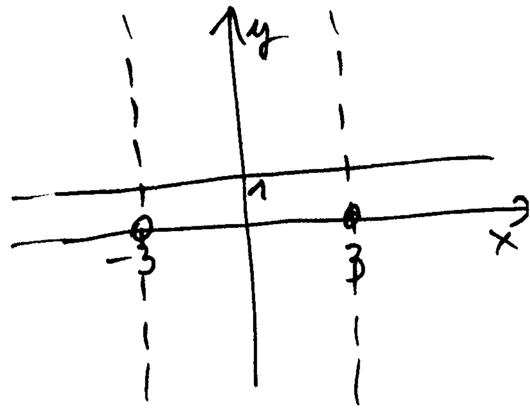
2) $f(x) = \frac{(x-2)^2}{x^2-9} = \frac{(x-2)^2}{(x-3)(x+3)}$



függőleges aszimptota: $\boxed{x=-3,\ x=3}$

mivel $\lim\limits_{-3\pm} \frac{(x-2)^2}{(x-3)(x+3)} = \frac{25}{-6}\cdot\frac{1}{0\pm} = \mp\infty$

$\lim\limits_{3\pm} \frac{(x-2)^2}{(x-3)(x+3)} = \frac{1}{6}\cdot\frac{1}{0\pm} = \pm\infty$

vízszintes aszimptota: $\boxed{y=1}$, mivel

$$\lim_{\infty} f(x) = \lim_{\infty} \frac{x^2-4x+4}{x^2-9} = \lim_{\infty} \frac{1-\frac{4}{x}+\frac{4}{x^2}}{1-\frac{9}{x^2}} = 1$$

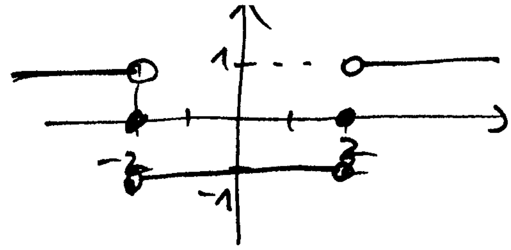
$g(x) = \frac{x(x+2)}{x-1}$

függőleges aszimptota: $\boxed{x=1}$, $\lim\limits_{1\pm} \frac{x(x+2)}{x-1} = \frac{3}{0\pm} = \pm\infty$

ferde aszimptota: $a = \lim\limits_{\infty} \frac{f(x)}{x} = \lim\limits_{\infty} \frac{x^2+2x}{x^2-x} = \lim\limits_{\infty} \frac{1+\frac{2}{x}}{1-\frac{1}{x}} = 1$

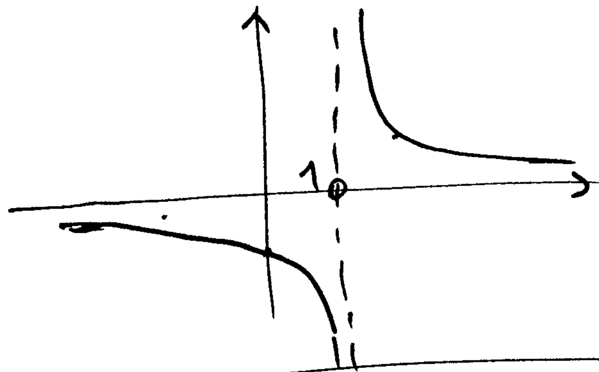
$b = \lim\limits_{\infty}(f(x)-x) = \lim\limits_{\infty}\left(\frac{x^2+2x}{x-1} - x\right) = \lim\limits_{\infty} \frac{3x}{x-1} = 3 \qquad \boxed{y = x+3}$

(3) $f(x)=\operatorname{sgn}(x^2-4)=\begin{cases}1, & |x|>2\\ 0, & |x|=2\\ -1, & |x|<2\end{cases}$



$x=-2$, $x=2$ pontokban ugrás van

$g(x)=\dfrac{1}{x-1}$



$x=1$-ben másodfajú szakadás van

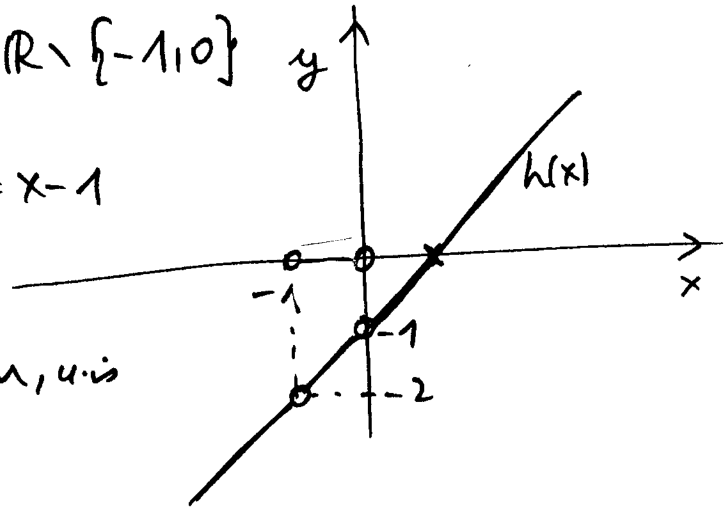
$h(x)=\dfrac{x(x-1)(x+1)}{x(x+1)}$, $\acute{E}_{h}=\mathbb{R}\setminus\{-1,0\}$

ha $x\neq 0$, $x\neq -1$ akkor $h(x)=x-1$

$x=-1$, $x=0$ -ban h-nak megszüntethető szakadása van, u.is

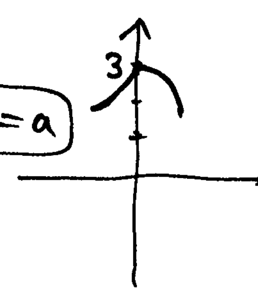
$\lim\limits_{0} h(x)=\lim\limits_{0}(x-1)=-1$

$\lim\limits_{-1} h(x)=\lim\limits_{-1}(x-1)=-2$



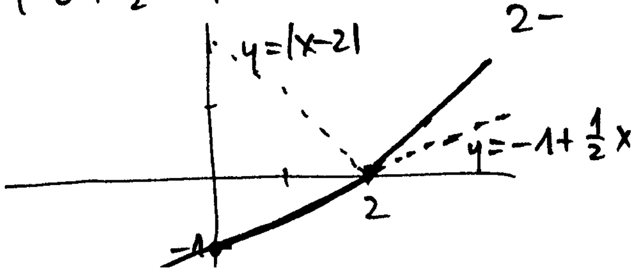
(4) $f(x)=\begin{cases}3e^{2x}, & x\le 0\\ a-x^3, & x>0\end{cases}$

$\left.\begin{array}{l} f(0)=3e^{0}=3\\ \lim\limits_{0+} f(x)=\lim\limits_{0+} 3e^{2x}=3\\ \lim\limits_{0-} f(x)=\lim\limits_{0-}(a-x^3)=a\end{array}\right\}\ \boxed{3=a}$



$g(x)=\begin{cases}|x-2|, & x\ge 2\\ b+\frac{1}{2}x, & x<2\end{cases}$

$\left.\begin{array}{l} f(2)=0=\lim\limits_{2+} f(x)=0\\ \lim\limits_{2-} f(x)=\lim\limits_{2-}(b+\frac{1}{2}x)=b+1\end{array}\right\}\ \begin{array}{l}0=b+1\\ \boxed{b=-1}\end{array}$



⑤ $f(x)=x^4-x^2-9$, f mindenhol folytonos

$$\left.\begin{array}{l} f(1)=1-1-9=-9<0 \\ f(2)=16-4-9=3>0 \end{array}\right\} \Rightarrow f\text{-nek } (1,2)\text{-ben van gyöke}$$

$x_1:=\frac{1+2}{2}=1{,}5$, $f(1{,}5)=-6{,}1875$ $\Rightarrow$ $(1{,}5;2)$-ben van gyök

$x_2:=\frac{2+1{,}5}{2}=1{,}75$, $f(1{,}75)\approx-3{,}246$ $\Rightarrow$ $(1{,}75;2)$-ben van gyök

$x_3:=\frac{2+1{,}75}{2}=1{,}875$, $f(1{,}875)\approx-0{,}256$ $\Rightarrow$ $(1{,}875;2)$-ben van gyök

$x_4:=\frac{1{,}875+2}{2}=1{,}9375$, $f(1{,}9375)\approx1{,}3378>0$

Tehát $(1{,}875;1{,}9375)$-ben van gyök

$x_5:=\frac{1{,}875+1{,}9375}{2}=1{,}90625$, $f(1{,}9)\approx0{,}4221>0$

Tehát $(1{,}875;1{,}9)$-ben van gyök

Mivel $1{,}9-1{,}875=0{,}025<0{,}07$, ezért ha az ismeretlen gyököt x_5-tel közelítjük, akkor a hiba $<\frac{7}{100}$.

Az ismeretlen gyök benne van az $(1{,}875;1{,}9)$ intervallumban! A megfelelő közelítés: $x^*\approx x_5=1{,}9$

V. ① a) $\left(3x^8-4x^5+2\cdot x^{\frac{2}{3}}+2\cdot x^{-\frac{1}{2}}\right)'=24x^7-20x^4+\frac{4}{3}x^{-\frac{1}{3}}-x^{-\frac{3}{2}}$

b) $\left[(4+3x)^8+e^{3x}\cdot x^2\right]'=8(4+3x)^7\cdot3+3e^{3x}\cdot x^2+e^{3x}\cdot2x$

c) $\left(\frac{2x^3-x^2}{x^4+x}\right)'=\left(\frac{2x^2-x}{x^3+1}\right)'=\frac{(4x-1)(x^3+1)-3x^2(2x^2-x)}{(x^3+1)^2}$

d) $\left(\frac{x\ln(x^2+1)}{\sin 2x}\right)' = \frac{\left[\ln(x^2+1)+x\frac{2x}{x^2+1}\right]\sin 2x - x\ln(x^2+1)2\cos 2x}{(\sin 2x)^2}$

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e) $\left(e^{x^2}+5^x+e^{x\ln x}\right)' = 2xe^{x^2}+5^x\cdot\ln 5+e^{x\ln x}(\ln x+1)$

$\left(x^x=(e^{\ln x})^x=e^{x\ln x}\right)$

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2. a) $f(0)=0$

$f'(x)=\left(xe^{-x^2}\right)'=e^{-x^2}+xe^{-x^2}\cdot(-2x)=e^{-x^2}(1-2x^2)$

$f'(0)=1$

$y=0+1\cdot(x-0)=x$ érintő egyenlete

$f(\bar{x})\approx y(\tilde{x}) \quad \Leftrightarrow f(-0{,}01)\approx -0{,}01=y(-0{,}01)$

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b) $f(3)=\frac{\sqrt{27-3}}{4}=\frac{\sqrt{6}}{2}$, $f'(x)=\frac{\frac{1}{2}\frac{3x^2-1}{\sqrt{x^3-x}}(x+1)-\sqrt{x^3-x}}{(x+1)^2}$

$f'(3)=\frac{13-12\sqrt{6}}{96}\approx -0{,}237$

$y=\frac{\sqrt{6}}{2}-0{,}237(x-3)$

$f(\bar{x})\approx y(\bar{x})=\frac{\sqrt{6}}{2}-0{,}237\cdot 0{,}02\approx 1{,}22$

VI. a) $f(x)=\frac{1}{4}x^3-3x=x\left(\frac{x^2}{4}-3\right)$

0. ÉT $f=\mathbb{R}$

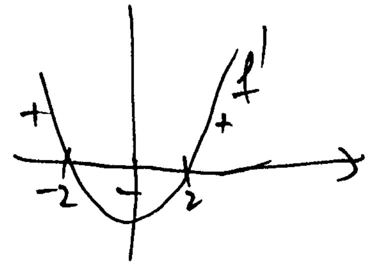
zh: $x=0$, $x=-\sqrt{12}\approx -3{,}44$, $x=\sqrt{12}\approx 3{,}44$

1. $f'(x)=\frac{3}{4}x^2-3=\frac{3}{4}(x^2-4)=\frac{3}{2}(x-2)(x+2)$, lehets. lok. szélsőérték helyei:

$x=2$, $x=-2$

1. tábl.

x	$(-\infty,-2)$	-2	$(-2,2)$	2	$(2,+\infty)$
f' előjele	+	0	−	0	+
f mon.	↗	lok. max	↘	lok. min.	↗



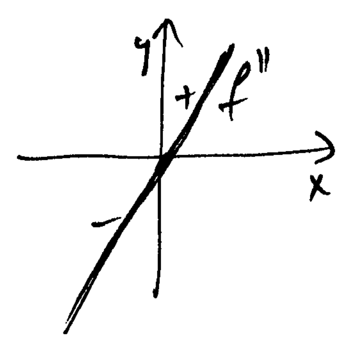
$f(-2)=4$ $f(2)=-4$

2.l.) $f''(x) = \frac{3}{2}x$ lehets. inflexiós hely: $x = 0$

2. tábl.

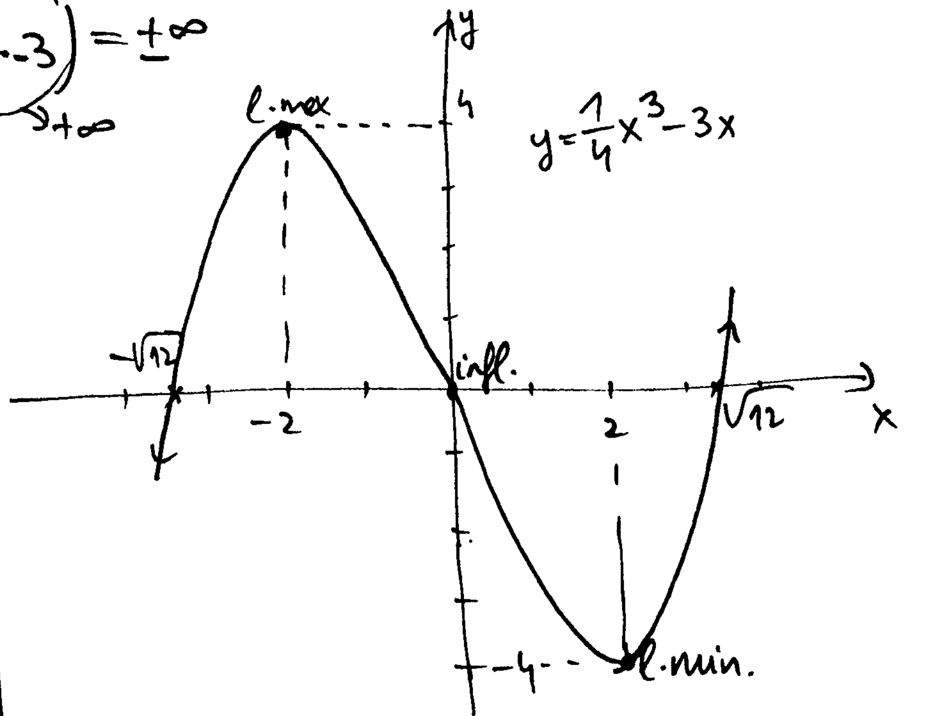
x	$(-\infty, 0)$	0	$(0, +\infty)$
f'' előjele	$-$		$+$
f görbülete	☹	infl.	☺

$f(0) = 0$



3.l.) $\lim\limits_{\pm\infty} f = \lim\limits_{\pm\infty} x\left(\frac{x^2}{4} - 3\right) = \pm\infty$, $\left(\frac{x^2}{4} - 3\right) \to +\infty$

aszimptota nincs



ⓑ $f(x) = \frac{x^2 - 9}{x^3}$ páratlan fv.

0.) $É.t. f = \mathbb{R} \setminus \{0\}$

zh: $x = 3$, $x = -3$

1.l.) $f'(x) = \frac{2x^4 - 3x^2(x^2 - 9)}{x^6}$

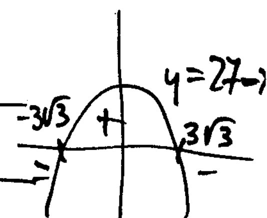
$f'(x) = \frac{27 - x^2}{x^4}$, lehets. lok. sz.é. helyek: $3\sqrt{3}$, $-3\sqrt{3} \approx -5,2$

$-3\sqrt{3}$ 0 $3\sqrt{3}$

1. tábl.

x	$(-\infty, -3\sqrt{3})$	$-3\sqrt{3}$	$(-3\sqrt{3}, 0)$	0	$(0, 3\sqrt{3})$	$3\sqrt{3}$	$(3\sqrt{3}, +\infty)$
f' előjele	$-$	0	$+$	nem $\exists$	$+$	0	$-$
f mon.	↘	l.min.	↗		↗	l.max.	↘

$f(-\sqrt{27}) \approx -0,2$ $f(\sqrt{27}) \approx 0,2$



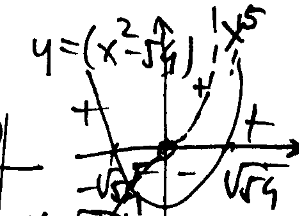
2.l.) $f''(x) = \frac{-2x^5 - 4x^3(27 - x^2)}{x^8} = \frac{2(x^2 - 54)}{x^5}$, lehets. inflexiós helyek: $\pm\sqrt{54}$

azaz $\pm 7,3$

$-\sqrt{54}$ 0 $\sqrt{54}$

2. tábl.

x	$(-\infty, -\sqrt{54})$	$-\sqrt{54}$	$(-\sqrt{54}, 0)$	0	$(0, \sqrt{54})$	$\sqrt{54}$	$(\sqrt{54}, +\infty)$
f'' előjele	$-$	0	$+$	$\nexists$	$-$	0	$+$
f görb.	⌒	infl.	☺		⌒	infl.	☺



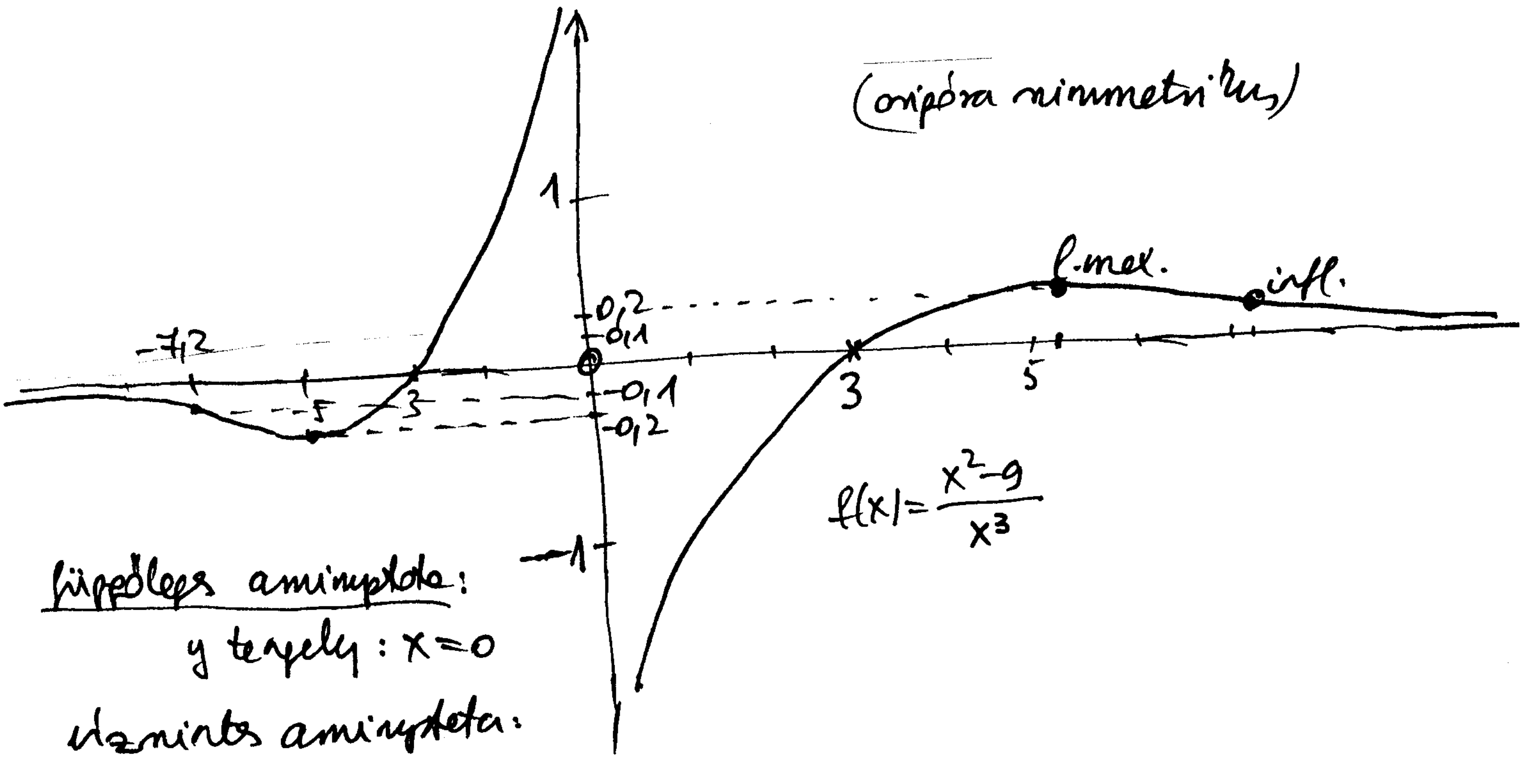
$f(-\sqrt{54}) = -0,11$

$f(\sqrt{54}) = 0,11$

3.l. $\lim\limits_{0-} \frac{x^2-9}{x^3} = \frac{-9}{0-} = +\infty$, $\lim\limits_{0+} \frac{x^2-9}{x^3} = \frac{-9}{0+} = -\infty$

$$\lim_{\infty} \frac{x^2-9}{x^3} = \lim_{\infty} \frac{\frac{1}{x} - \frac{9}{x^3}}{1} = 0$$

(origóra szimmetrikus)



függőleges aszimptota:
y tengely: $x=0$

vízszintes aszimptota:
x tengely: $y=0$

c) $f(x) = -4x^5 + 15x^3$ $\acute{E}t\, f = \mathbb{R}$, zh: $x^3(15-4x^2)=0 \Rightarrow$

$x=0$, $x = \frac{\sqrt{15}}{2}$, $x = -\frac{\sqrt{15}}{2} \approx -1{,}936$ (f páratlan fv.)

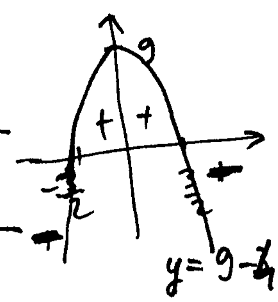
$f'(x) = -20x^4 + 45x^2 = 5x^2(-4x^2+9) = 5x^2(3-2x)(3+2x)$

lehets. lok. sz. é. helyek: $x=0$, $x=-3/2$, $x=\frac{3}{2}=1{,}5$

1. tábl.

x	$(-\infty, -3/2)$	$-\frac{3}{2}$	$(-\frac{3}{2}, 0)$	0	$(0, 3/2)$	$3/2$	$(\frac{3}{2}, +\infty)$
f' előjele	−	0	+	0	+	0	−
f mon.	↘	l.min.	↗		↗	l.max.	↘

$f(-\frac{3}{2}) = -20{,}25$ $f(\frac{3}{2}) = 20{,}25$



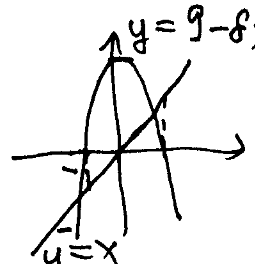
$f''(x) = -80x^3 + 90x = 10x(9-8x^2)$

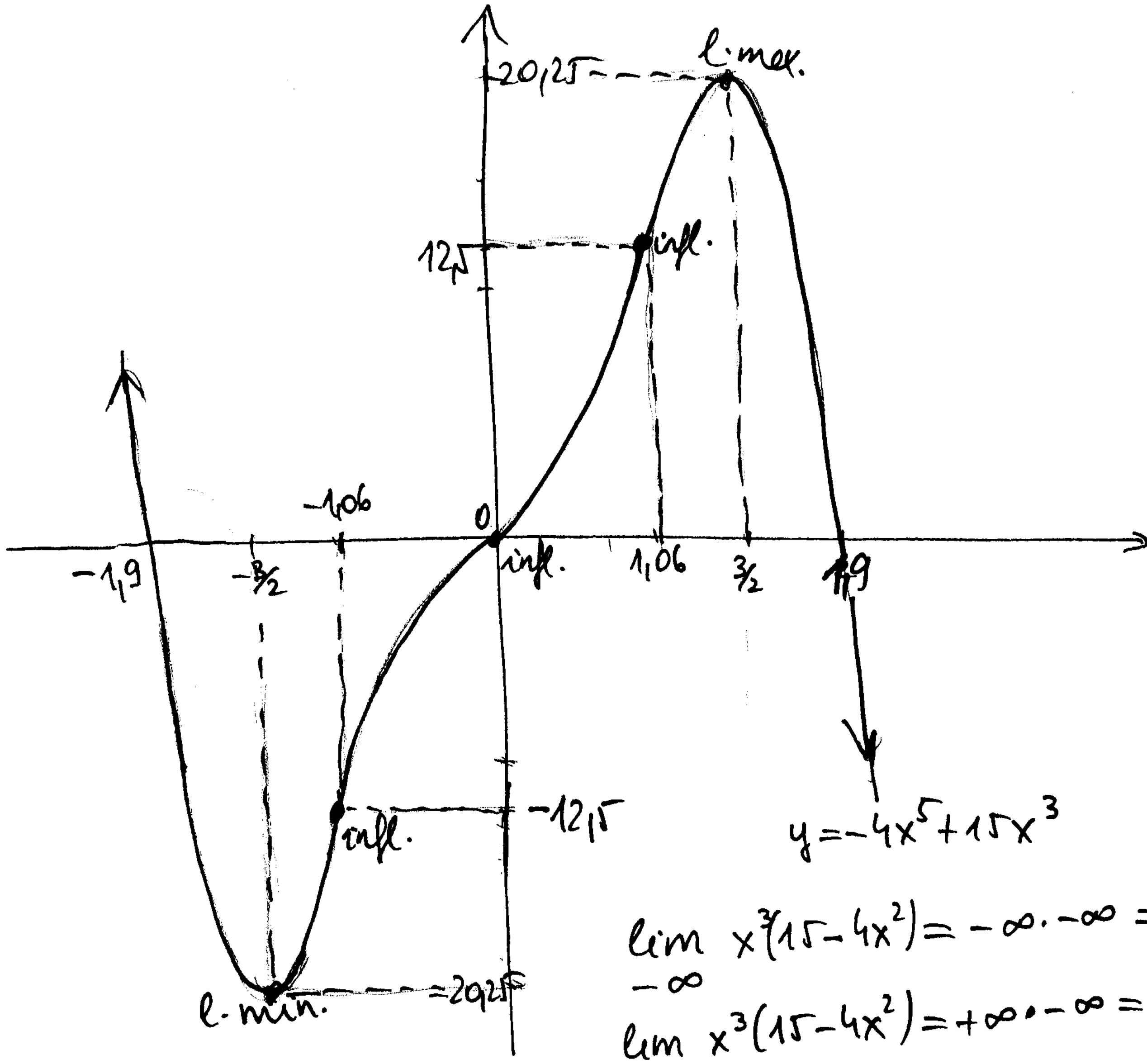
lehets. inflexiós helyek: $x=0$, $x=-\frac{3\sqrt{2}}{4}$, $x=\frac{3\sqrt{2}}{4} \approx 1{,}06$

2. tábl.

x	$(-\infty, -\frac{3\sqrt{2}}{4})$	$-\frac{3\sqrt{2}}{4}$	$(-\frac{3\sqrt{2}}{4}, 0)$	0	$(0, \frac{3\sqrt{2}}{4})$	$\frac{3\sqrt{2}}{4}$	$(\frac{3\sqrt{2}}{4}, +\infty)$
f'' előjele	+	0	−	0	+	0	−
f görbülete	∪	infl.	∩	infl.	∪	infl.	∩

$f(-\frac{3\sqrt{2}}{4}) = -12{,}5$ $f(0)=0$ $f(\frac{3\sqrt{2}}{4}) = 12{,}5$





$$y=-4x^5+15x^3$$

$$\lim_{-\infty} x^3(15-4x^2)=-\infty\cdot-\infty=+\infty$$

$$\lim_{+\infty} x^3(15-4x^2)=+\infty\cdot-\infty=-\infty$$

assimptota nincs!

d) $f(x)=\dfrac{2x+1}{(x+1)^2}$ Ét $f=\mathbb{R}\setminus\{-1\}$, zh: $x=-\frac{1}{2}$

$$f'(x)=\frac{2(x+1)^2-2\cancel{(x+1)}(2x+1)}{(x+1)^{\cancel{4}\,3}}=\frac{-2x}{(x+1)^3}$$

, lehets. lok. sz. é. hely: $x=0$

1. táбл.

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,+\infty)$
f' előjele	$-$	nem ért.	$+$	0	$-$
f mon.	$\searrow$		$\nearrow$	l.max.	$\searrow$

$f(0)=1$

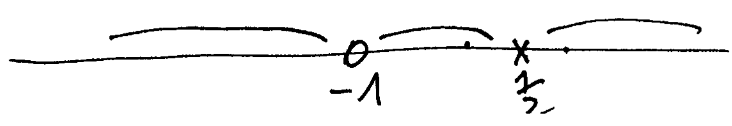
$$f'(-2)=\frac{4}{-1}=-4<0$$

$$f'(-\tfrac{1}{2})=\frac{1}{(\frac{1}{2})^3}>0$$

$$f'(1)=\frac{-2}{8}<0$$

$$f''(x)=\frac{-2(x+1)^3-3(x+1)^2\cdot(-2x)}{(x+1)^6}=\frac{-2\cancel{(x+1)^2}\left[x+1-3x\right]}{(x+1)^{\cancel{6}\,4}}=\frac{-2(1-2x)}{(x+1)^4}=\frac{2(2x-1)}{(x+1)^4}$$

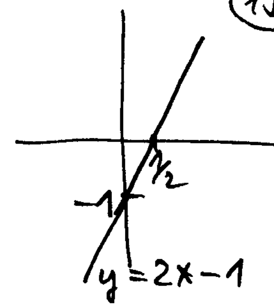
lehets. infl. hely: $x=\frac{1}{2}$



2. tábl.

x	$(-\infty,-1)$	-1	$(-1,\frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2},+\infty)$
f'' előjele	$-$	nem $\exists$	$-$	0	$+$
f görb.	$\frown$		$\frown$	infl.	$\smile$

$f(\frac{1}{2}) = \frac{8}{9} \approx 0{,}88$

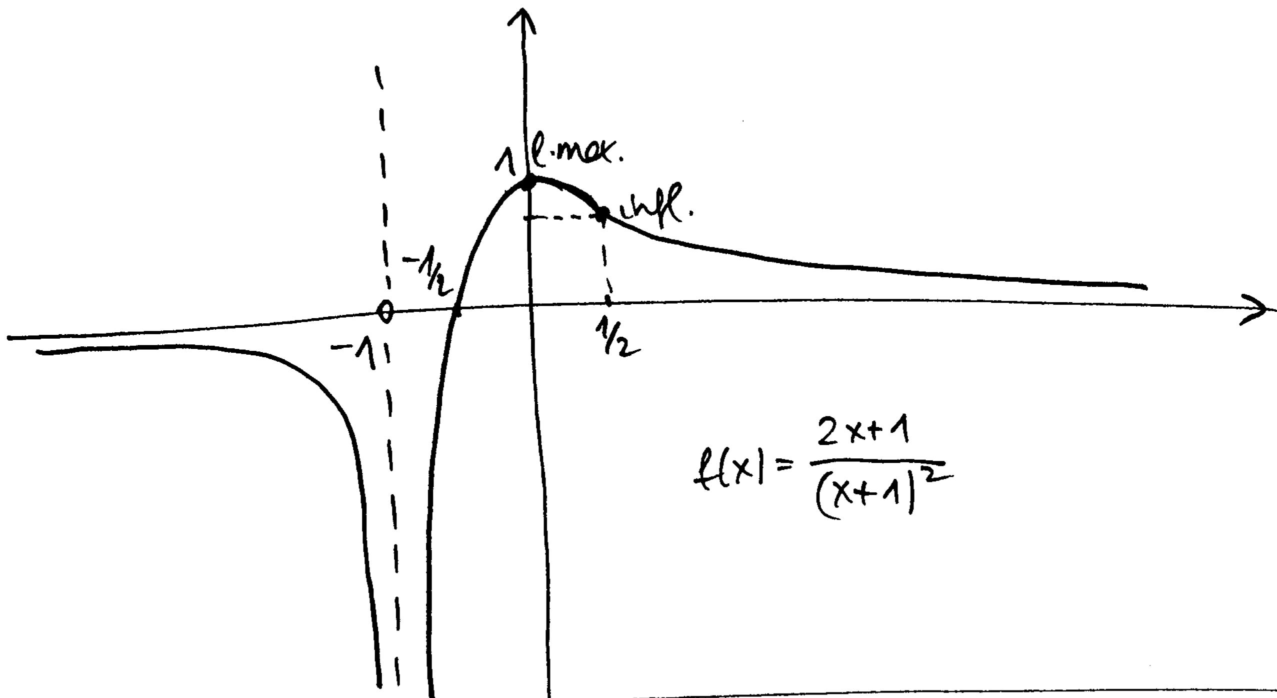


$$\lim_{-1} \frac{2x+1}{(x+1)^2} = \frac{-1}{0+} = -\infty$$

$$\lim_{\infty} \frac{2x+1}{x^2+2x+1} = \lim_{\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}} = \frac{0}{1} = 0$$

vízszintes aszimptota : x tengely $\Leftrightarrow y=0$

függőleges -"- : $x=-1$



VII. Integrálás

1) $\int \left(x^{\frac{7}{2}} + 2^x + \sin 3x + \frac{1}{x}\right) dx = \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + \frac{2^x}{\ln 2} - \frac{1}{3}\cos 3x + \ln|x| + c$

2) $\int \underset{g}{x}\,\underset{f'}{e^{2x}} \overset{\text{parc.}}{=} \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$\begin{pmatrix} g' = 1 \\ f = \frac{1}{2}e^{2x} \end{pmatrix}$

$\int \underset{g}{x}\,\underset{f'}{\sin 3x} \overset{\text{parc.}}{=} -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x = -\frac{1}{3}\cos 3x + \frac{1}{9}\sin 3x + c$

$\begin{pmatrix} g' = 1 \\ f = -\frac{1}{2}\cos 3x \end{pmatrix}$

$$\int \underset{f'}{x}\, \underset{g}{\ln x} \;\overset{\text{parc.}}{=}\; \frac{1}{2}x^2\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x} = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x =$$

$$\left(\begin{array}{l} f = \frac{x^2}{2} \\ g' = \frac{1}{x} \end{array}\right) \qquad = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$$

---

3) $$\int \frac{3}{x(x+1)} = \int\left(\frac{A}{x} + \frac{B}{x+1}\right) \overset{*}{=} \int\frac{3}{x} - \int\frac{3}{x+1} = 3\ln x - 3\ln(x+1) + c$$

$$= \ln\left(\frac{x}{x+1}\right)^3 + c$$

$$\left(\begin{array}{l} * \quad A(x+1) + Bx = 3 \quad \forall x \\ \left.\begin{array}{l} A + B = 0 \\ A = 3 \end{array}\right\} \; B = -3 \end{array}\right)$$

$$\int \frac{2x+1}{(x-1)(x+1)} = \int\left(\frac{A}{x-1} + \frac{B}{x+1}\right) \overset{*}{=} \frac{3}{2}\int\frac{1}{x-1} + \frac{1}{2}\int\frac{1}{x+1} =$$

$$\left(\begin{array}{l} * \quad A(x+1) + B(x-1) = 2x+1 \quad \forall x \Rightarrow \\ \left.\begin{array}{l} A + B = 2 \\ A - B = 1 \end{array}\right\} \; A = \frac{3}{2} \;,\; B = \frac{1}{2} \end{array}\right) \quad \begin{array}{l} = \frac{3}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) \\ = \ln\sqrt{(x-1)^3(x+1)} + c \end{array}$$

---

4) $$\int_1^2 (2x+1)^{10} = \left[\frac{(2x+1)^{11}}{11\cdot 2}\right]_{x=1}^{x=2} = \frac{1}{22}\left[(2x+1)^{11}\right]_{x=1}^{x=2} = \frac{1}{22}\left(5^{11} - 3^{11}\right)$$

$$\int_{-3}^{-1} \frac{4x+1}{x^3} = \int_{-3}^{-1}\left(4\cdot x^{-2} + x^{-3}\right) = \left[\frac{4x^{-1}}{-1} + \frac{x^{-2}}{-2}\right]_{x=-3}^{x=-1} = \left[-\frac{4}{x} - \frac{1}{2x^2}\right]_{x=-3}^{x=-1} =$$

$$= \left(4 - \frac{1}{2}\right) - \left(\frac{4}{3} - \frac{1}{18}\right) = \frac{20}{9} \approx 2{,}22$$

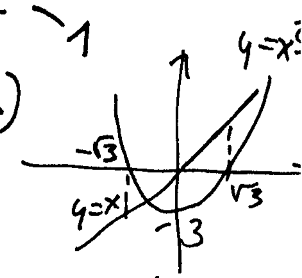
$$\int_1^4 \left(3x^4 - 2x^3 + 2\cdot x^{-\frac{1}{2}}\right) = \left[\frac{3x^5}{5} - 2\cdot\frac{x^4}{4} + 2\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{x=1}^{x=4} = \left[\frac{3x^5}{5} - \frac{x^4}{2} + 4\sqrt{x}\right]_{x=1}^{x=4} =$$

$$= \underbrace{\left(614{,}4 - 128 + 8\right)}_{494{,}4} - \underbrace{\left(\frac{3}{5} - \frac{1}{2} + 4\right)}_{4{,}1} = 490{,}3$$

5) $f(x)=x^4-6x^2+5$, páros fv.

a) zh: $(x^2)^2-6x^2+5=0$, $(x^2)_{1,2}=3\pm\sqrt{9-5}=3\pm 2$ ⟨ 5, 1

$x_1=-\sqrt{5}$, $x_2=\sqrt{5}$, $x_3=-1$, $x_4=1$ $(\sqrt{5}\approx 2{,}23)$



$f'(x)=4x^3-12x=4x(x^2-3)$

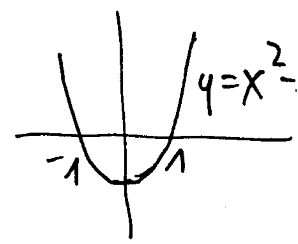
lehets. lok. sz. é. helyek: $x=0$, $x=-\sqrt{3}$, $x=\sqrt{3}$

1. tábl.

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},+\infty)$
f' előjele	$-$	0	$+$	0	$-$	0	$+$
f mon.	↘	l. min.	↗	l. max	↘	l. min.	↗

$f(-\sqrt{3})=-4$ $f(0)=5$ $f(\sqrt{3})=-4$

$f''(x)=12x^2-12=12(x^2-1)=12(x-1)(x+1)$



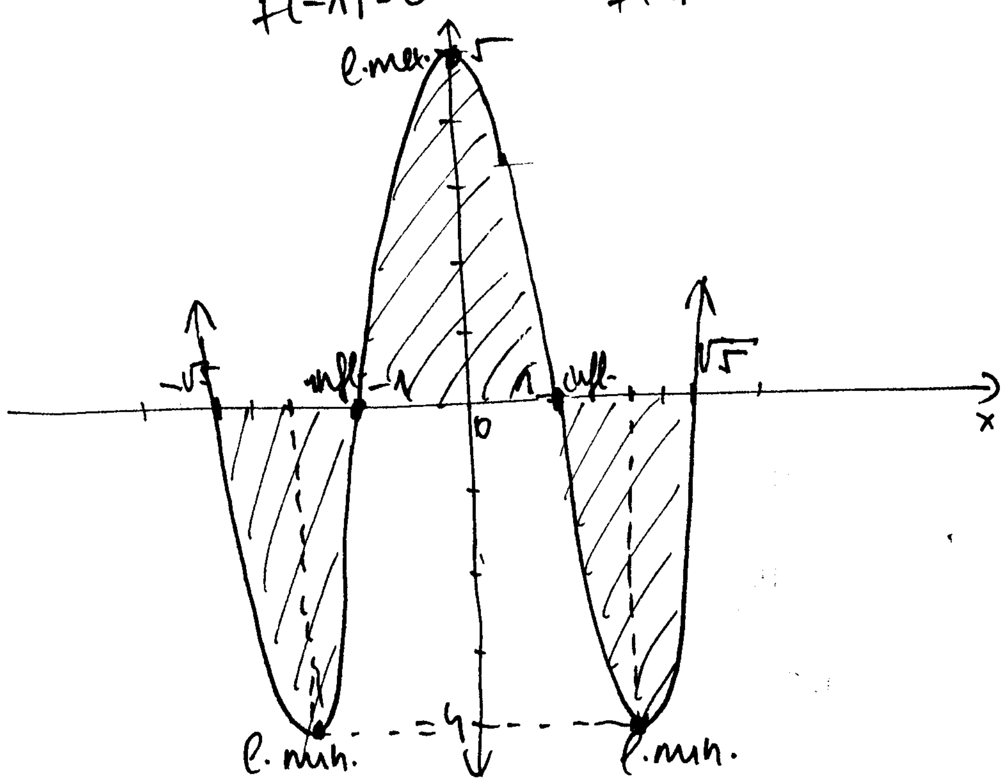
lehets. infl. helyek: $x=1$, $x=-1$

2. tábl.

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
f'' előjele	$+$	0	$-$	0	$+$
f görb.	◡	infl.	◠	infl.	◡

$f(-1)=0$ $f(1)=0$

f képe:



A besatírozott tartomány területét keressük. Mivel az y tengelyre szimmetrikus a tartomány, elég $x\geq 0$ esetet nézni.

$$T=2\left\{\int_0^1 (x^4-6x^2+5)\,dx+\int_1^{\sqrt5} -(x^4-6x^2+5)\,dx\right\}=$$

$$=2\left\{\left[\frac{x^5}{5}-\overset{2}{\cancel{6}}\frac{x^3}{\cancel{3}}+5x\right]_{x=0}^{x=1}+\left[-\frac{x^5}{5}+\overset{2}{\cancel{6}}\frac{x^3}{\cancel{3}}-5x\right]_{x=1}^{x=\sqrt5}\right\}=$$

$$=2\left\{\underbrace{\left(\frac15-2+5\right)}_{\frac{16}{5}}+\left(\underbrace{\left(-5\sqrt5+10\sqrt5-5\sqrt5\right)}_{0}\underbrace{-\left(-\frac15+2-5\right)}_{\frac{16}{5}}\right)\right\}=2\cdot\frac{32}{5}=$$

$$=\frac{64}{5}=12{,}8$$

5/b $f(x)=4x$

$g(x)=x^2+2x-3=(x+3)(x-1)$

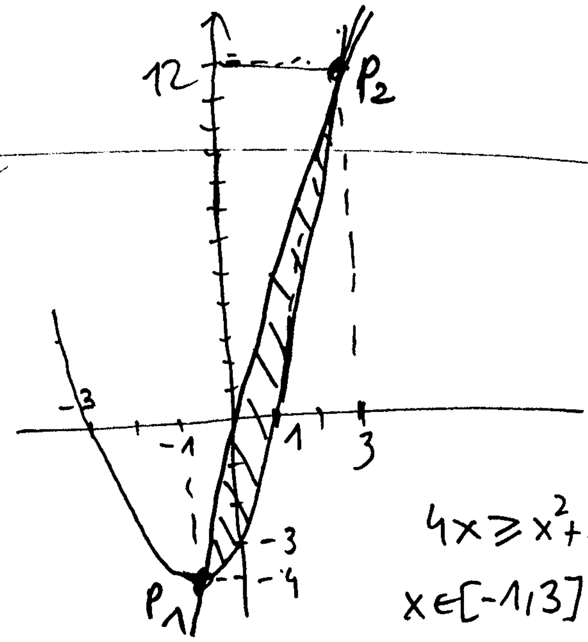
metszéspontok:

$4x=x^2+2x-3$

$x^2-2x-3=0$

$(x-3)(x+1)=0$, $x_1=-1,\ x_2=3$

$P_1(-1,-4)$ $P_2(3,12)$



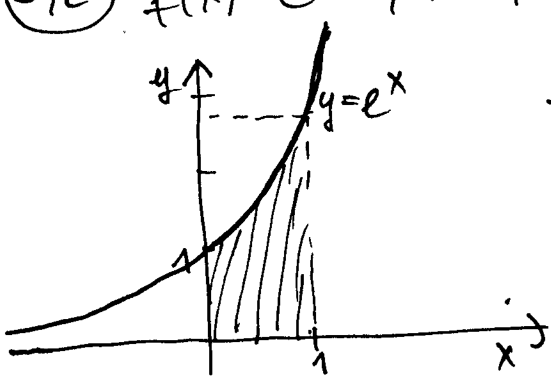
$4x\geq x^2+2x-3$

$x\in[-1,3]$

$$T=\int_{-1}^{3}\left(4x-(x^2+2x-3)\right)=\int_{-1}^{3}(2x+3-x^2)\,dx=\left[x^2+3x-\frac{x^3}{3}\right]_{x=-1}^{x=3}=$$

$$=\underbrace{(9+9-9)}_{9}-\underbrace{\left(1-3+\frac13\right)}_{-\frac53}=9+\frac53=\frac{32}{3}\approx10{,}67$$

5/c $f(x)=e^x$, $x=0,\ x=1$



$$T=\int_0^1 e^x=\left[e^x\right]_{x=0}^{x=1}=e-1\approx1{,}72$$

1) a, $y' = x(1+y^2)$, $y(0)=0$ | szétválasztható változójú

$\frac{dy}{dx} = x\cdot(1+y^2)$, $\int \frac{dy}{1+y^2} = \int x\,dx$

$\operatorname{arctg} y = \frac{1}{2}x^2 + c$, $y = tg\left(\frac{x^2}{2}+c\right)$

$y(0)=0 \Leftrightarrow 0 = tg\, c \Leftrightarrow c=0$; $y = tg\,\frac{x^2}{2}$

b, $y' - \frac{2}{x}\cdot y = -2 - \frac{2}{x}$, $y(1)=1$ | elsőrendű lin. inhom.

(h) $y' - \frac{2}{x}y = 0$ szétválasztható

$\frac{dy}{dx} = \frac{2}{x}\cdot y$, $\int \frac{dy}{y} = \int \frac{2}{x}\,dx$, $\ln y = \ln c\cdot x^2$ | $y_{h,a} = cx^2$

$y_p = c(x)\cdot x^2$ (állandó variálása)

$c'\cdot x^2 + \underbrace{\left(2xc - \frac{2}{x}\cdot c\cdot x^2\right)}_{0} = -2 - \frac{2}{x}$, $c' = -\frac{2}{x^2} - \frac{2}{x^3}$

$c = \int\left(-\frac{2}{x^2} - \frac{2}{x^3}\right) = \frac{2}{x} + \frac{1}{x^2}$; $y_p = \left(\frac{2}{x} + \frac{1}{x^2}\right)\cdot x^2 = 1+2x$

$y_{i,a} = cx^2 + 1 + 2x$

$y(1)=1 \Leftrightarrow 1 = c+1+2$, $c=-2$

Tehát $\boxed{y(x) = -2x^2 + 1 + 2x}$

c, $y' = e^x\cdot e^{2y}$, $y(0)=2$ | szétválasztható változójú diffe.

$\frac{dy}{dx} = e^x\cdot e^{2y}$, $\int e^{-2y}\,dy = \int e^x\,dx$

$-\frac{1}{2}e^{-2y} = e^x + c$

$e^{-2y} = c - 2e^x$

$-2y = \ln(c-2e^x)$

$y = -\frac{1}{2}\ln(c-2e^x)$

$y(0)=2 \Leftrightarrow 2 = -\frac{1}{2}\ln(c-2)$

$-4 = \ln(c-2)$

$e^{-4} = c-2$

$c = 2+e^{-4}$

$y(x) = -\frac{1}{2}\ln\left(2+e^{-4}-2e^x\right)$

d) $y' + 2y = x$, $y(0) = \frac{3}{4}$ | inhom., lineáris

1) (h) $y' + 2y = 0$ szétválasztható

$\frac{dy}{dx} = -2y$, $\int \frac{dy}{y} = -\int 2\,dx$, $\ln y = -2x + c$

$y_{h,a} = ce^{-2x}$ 2.) $y_p = c(x) \cdot e^{-2x}$ beírjuk az eredeti diff.e.-be:

$c' e^{-2x} \underbrace{-2ce^{-2x} + 2ce^{-2x}}_{0} = x$, $c' = xe^{-2x}$

$c(x) = \int x e^{+2x} \xrightarrow{\text{parc. int.}}_{\substack{g'=1 \\ f = +\frac{1}{2}e^{+2x}}} +\frac{1}{2}xe^{+2x} - \frac{1}{2}\int e^{+2x} = +\frac{1}{2}xe^{+2x} - \frac{1}{4}e^{+2x}$

$y_p = e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right) e^{-2x} = \frac{x}{2} - \frac{1}{4}$

$y_{i,a} = c \cdot e^{-2x} + \frac{x}{2} - \frac{1}{4}$

$y(0) = \frac{3}{4} \Leftrightarrow \frac{3}{4} = c - \frac{1}{4}$, $c = 1$

Tehát a feladat megoldása: $\boxed{y = e^{-2x} + \frac{x}{2} - \frac{1}{4}}$

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e) $y' = x \cdot e^{-x} \cdot \frac{1}{y^3 + \sin 2y}$ | szétválasztható változójú:

$\frac{dy}{dx} = \frac{xe^{-x}}{y^3 + \sin 2y}$, $\int (y^3 + \sin 2y)\,dy = \int xe^{-x}\,dx$

$\int (y^3 + \sin 2y)\,dy = \frac{y^4}{4} - \frac{1}{2}\cos 2y + c$

$\int x e^{-x}\,dx \xrightarrow{\text{parc. int.}}_{\substack{g'=1 \\ f = -e^{-x}}} -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + c$

$\frac{y^4}{4} - \frac{1}{2}\cos 2y = C - e^{-x}(x+1)$ (implicit egyenlet y-ra)

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f) $y' - \frac{1}{x}y = 3x^3 - x$, $y(1) = 2$ | lin., inhom.

1) (h) $y' - \frac{1}{x}y = 0$, $\frac{dy}{dx} = \frac{y}{x}$, $\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln y = \ln c \cdot x$, $y_{h,a} = c \cdot x$

2) $y_p = c(x) \cdot x$ beírjuk az eredeti diff.e.-be:

$c'x + c - \frac{1}{x}\cdot c\cdot x = 3x^3 - x$

$c' = 3x^2 - 1$, $c(x) = \int (3x^2-1)\,dx = x^3 - x$ $\Big\}$ $y_p = (x^3-x)\cdot x = x^4 - x^2$

$y_{i,a} = cx + x^4 - x^2$

$y(1) = 2 \iff 2 = c + 1 - 1 = c$, $c = 2$

Tehát $\boxed{y(x) = 2x + x^4 - x^2}$

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2) a) $y'' - 6y' + 5y = 2x^2$

(h) $\lambda^2 - 6\lambda + 5 = 0 = (\lambda-1)(\lambda-5)$, $\lambda_1 = 1$, $\lambda_2 = 5$

$y_{h,a} = c_1 e^x + c_2 e^{5x}$

$y_p = Ax^2 + Bx + C$
$y_p' = 2Ax + B$
$y_p'' = 2A$

$2A - 12Ax - 6B + 5Ax^2 + 5Bx + 5C = 2x^2$

x^2 együtthatói : $5A = 2$
x –"– : $-12A + 5B = 0$
konstansok : $2A - 6B + 5C = 0$

$\rightarrow A = \frac{2}{5}$, $B = \frac{24}{25}$, $C = \frac{124}{125}$

$y_p = \frac{2}{5}x^2 + \frac{24}{25}x + \frac{124}{125}$

Tehát $\boxed{y_{i,a} = c_1 e^x + c_2 e^{5x} + \frac{2}{5}x^2 + \frac{24}{25}x + \frac{124}{125}}$

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b) $y'' - 7y' + 6y = \cos 2x$

(h) $\lambda^2 - 7\lambda + 6 = 0 = (\lambda-1)(\lambda-6)$, $\lambda_1 = 1$, $\lambda_2 = 6$

$y_{h,a} = c_1 e^x + c_2 e^{6x}$

$y_p = A\cos 2x + B\sin 2x$
$y_p' = -2A\sin 2x + 2B\cos 2x$
$y_p'' = -4A\cos 2x - 4B\sin 2x$

$-4A\cos 2x - 4B\sin 2x + 14A\sin 2x - 14B\cos 2x$
$+ 6A\cos 2x + 6B\sin 2x = \cos 2x$

$2A - 14B = 1$ $\quad$ $A - 7B = \frac{1}{2}$
$2B + 14A = 0$ $\quad$ $B + 7A = 0$, $B = -7A$

$A + 49A = \frac{1}{2}$, $50A = \frac{1}{2}$, $A = \frac{1}{100}$, $B = -\frac{7}{100}$

$y_p = \frac{1}{100}\cos 2x - \frac{7}{100}\sin 2x$. $\boxed{y_{i,a} = c_1 e^x + c_2 e^{6x} + 0{,}01\cos 2x - 0{,}07\sin 2x}$

c) $\boxed{y'' - 3y' - 10y = 3e^{4x}}$

(h) $\lambda^2 - 3\lambda - 10 = 0 = (\lambda - 5)(\lambda + 2)$, $\lambda_1 = 5$, $\lambda_2 = -2$

$y_{h,a} = c_1 e^{5x} + c_2 e^{-2x}$

$$\left.\begin{aligned} y_p &= A e^{4x} \\ y_p' &= 4Ae^{4x} \\ y_p'' &= 16Ae^{4x} \end{aligned}\right\} \quad 16Ae^{4x} - 12Ae^{4x} - 10Ae^{4x} = 3e^{4x}$$

$$-6A = 3$$

$$A = -\frac{1}{2}$$

$y_p = -\frac{1}{2} e^{4x}$

$$\boxed{y_{i,a} = c_1 e^{5x} + c_2 e^{-2x} - \frac{1}{2} e^{4x}}$$

Vége!

Vizsgára kell:

I. /4
II.
III. /1
IV. /1, 4,
! V. /1, 2
VI.
VII. 1, -5, b, c
