Gyakorló feladatok 2.

Cholesky felbontás, Householder felbontás

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1. Határozzuk meg az

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix} \qquad \text{mátrix Cholesky felbontását!}$$

Emlékeztető :

$A = L \cdot L^T$, ahol L alsó $\triangle$ mátrix.

$$a_{ij} = \sum_{k=1}^{j} l_{ik}\, l_{jk}\,, \qquad a_{ii} = \sum_{k=1}^{i} l_{ik}^2$$

Megoldás

$$a_{11} = l_{11}^2 \implies 4 = l_{11}^2 \implies \boxed{l_{11} = 2}$$

$$a_{21} = l_{21} \cdot l_{11} \implies -1 = l_{21} \cdot 2 \implies \boxed{l_{21} = -\frac{1}{2}}$$

$$a_{31} = l_{31} \cdot l_{11} \implies 1 = l_{31} \cdot 2 \implies \boxed{l_{31} = \frac{1}{2}}$$

$$a_{22} = l_{21}^2 + l_{22}^2 \implies \frac{17}{4} = \frac{1}{4} + l_{22}^2 \implies 4 = l_{22}^2 \implies \boxed{l_{22} = 2}$$

$$a_{32} = l_{31} \cdot l_{21} + l_{32} \cdot l_{22} \implies \frac{11}{4} = \frac{1}{2} \cdot -\frac{1}{2} + l_{32} \cdot 2 \implies \boxed{l_{32} = \frac{3}{2}}$$

$$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies \frac{7}{2} = \frac{1}{4} + \frac{9}{4} + l_{33}^2 \implies 1 = l_{33}^2 \implies \boxed{l_{33} = 1}$$

Tehát : $$L = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & 2 & 0 \\ 1/2 & 3/2 & 1 \end{bmatrix}$$

Ellenőrzés

$$\begin{bmatrix} 2 & 0 & 0 \\ -1/2 & 2 & 0 \\ 1/2 & 3/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1/2 & 1/2 \\ 0 & 2 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

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2. Oldjuk meg Cholesky felbontással a következő egyenletrendszert:

$$\begin{array}{rcrcrcl} 4x_1 & - & x_2 & & & = & 2 \\ -x_1 & + & 4x_2 & - & x_3 & = & 6 \\ & & -x_2 & + & 4x_3 & = & 2 \end{array}$$

Megoldás

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}, \qquad \underline{\mathbf{b}} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

Először hozzuk A - t $L \cdot L^T$ alakra, ahol L alsó $\triangle$ mátrix.

$$a_{11} = l_{11}^2 \quad \Longrightarrow \quad \boxed{l_{11} = 2}$$

$$a_{21} = l_{21} \cdot l_{11} \quad \Longrightarrow \quad \boxed{l_{21} = -\frac{1}{2}}$$

$$a_{31} = l_{31} \cdot l_{11} \quad \Longrightarrow \quad \boxed{l_{31} = 0}$$

$$a_{22} = l_{21}^2 + l_{22}^2 \quad \Longrightarrow \quad \boxed{l_{22} = \frac{\sqrt{15}}{2}}$$

$$a_{32} = l_{31} \cdot l_{21} + l_{32} \cdot l_{22} \quad \Longrightarrow \quad \boxed{l_{32} = -\frac{2\sqrt{15}}{15}}$$

$$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2 \quad \Longrightarrow \quad \boxed{l_{33} = \frac{2\sqrt{210}}{15}}$$

$$\Longrightarrow \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2\sqrt{15}/15 & 2\sqrt{210}/15 \end{bmatrix}$$

Az $LL^T\underline{\mathbf{x}} = \underline{\mathbf{b}}$ egyenletrendszert két lépésben oldjuk meg:

1. $\quad L\underline{\mathbf{y}} = \underline{\mathbf{b}}$

2. $\quad L^T\underline{\mathbf{x}} = \underline{\mathbf{y}}$

1.

			y_1 y_2 y_3
2	0	0	2
-1/2	$\sqrt{15}/2$	0	6
0	-2$\sqrt{15}/15$	$2\sqrt{210}/15$	2

$$\boxed{y_1 = 1}$$

$$\Longrightarrow \qquad -\frac{1}{2} + \frac{\sqrt{15}}{2} \cdot y_2 = 6 \qquad \Longrightarrow \qquad \boxed{y_2 = \frac{13\sqrt{15}}{15}}$$

$$-\frac{2\sqrt{15}}{15} \cdot \frac{13\sqrt{15}}{15} + \frac{2\sqrt{210}}{15} \cdot y_3 = 2 \qquad \Longrightarrow \qquad \boxed{y_3 = \frac{2\sqrt{210}}{15}}$$

$$\Longrightarrow \qquad \underline{\mathbf{y}} = \begin{bmatrix} 1 \\ \frac{13\sqrt{15}}{15} \\ \frac{2\sqrt{210}}{15} \end{bmatrix}$$

2.

$$\begin{array}{ccc|c} & & & x_1 \\ & & & x_2 \\ & & & x_3 \\ \hline 2 & \text{-1/2} & 0 & 1 \\ 0 & \sqrt{15}/2 & -2\sqrt{15}/15 & 13\sqrt{15}/15 \\ 0 & 0 & 2\sqrt{210}/15 & 2\sqrt{210}/15 \end{array}$$

$$\frac{2\sqrt{210}}{15} \cdot x_3 = \frac{2\sqrt{210}}{15} \quad \Longrightarrow \quad \boxed{x_3 = 1}$$

$$\Longrightarrow \quad \frac{\sqrt{15}}{2} \cdot x_2 - \frac{2\sqrt{15}}{15} = \frac{13\sqrt{15}}{15} \quad \Longrightarrow \quad \boxed{x_2 = 2}$$

$$2 \cdot x_1 + \left(-\frac{1}{2}\right) \cdot 2 = 1 \quad \Longrightarrow \quad \boxed{x_1 = 1}$$

$$\Longrightarrow \quad \underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Ellenőrzés

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

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3. Határozzuk meg az

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix} \qquad \text{mátrix } QR \text{ felbontását!}$$

Emlékeztető :

$$\mathbb{R}^{n\times n} \ni A = QR \quad \Longrightarrow \quad H^{n-1} \cdot \ldots \cdot H^1 A = R \quad \Longrightarrow \quad A = \underbrace{\left(\left(H^1\right)^{-1} \cdot \ldots \cdot \left(H^{n-1}\right)^{-1} \right)}_{Q} R$$

$$H = I - 2\underline{\mathbf{h}}\,\underline{\mathbf{h}}^T, \qquad \underline{\mathbf{h}} = \frac{\underline{\mathbf{x}} - \sigma\,\underline{\mathbf{e}}_1}{\|\,\underline{\mathbf{x}} - \sigma\,\underline{\mathbf{e}}_1\,\|_2}, \qquad |\sigma| = \|\,\underline{\mathbf{x}}\,\|_2\,, \qquad \sigma := -\mathbf{sign}\,(x_1)\,\|\,\underline{\mathbf{x}}\,\|_2$$

Megoldás

$$\underline{\mathbf{x}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad \|\,\underline{\mathbf{x}}\,\|_2 = \sqrt{9+16} = 5\,, \qquad \sigma = -5\,,$$

$$\underline{\mathbf{x}} - \sigma\,\underline{\mathbf{e}}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix},$$

$$\|\,\underline{\mathbf{x}} - \sigma\,\underline{\mathbf{e}}_1\,\|_2 = \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\|_2 = \sqrt{64+16} = 4\sqrt{5}\,,$$

$$\underline{\mathbf{h}} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix}, \qquad \underline{\mathbf{h}}\,\underline{\mathbf{h}}^T = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} \cdot \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 2/5 \end{bmatrix} = \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix} = Q^{-1} = Q^T = Q\,,$$

$$HA = \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -12/5 \\ 0 & -16/5 \end{bmatrix} = R\,.$$

Ellenőrzés

$$QR = \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} -5 & -12/5 \\ 0 & -16/5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix} = A$$

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GYAKORLÓ FELADATOK

1. Határozzuk meg a következő mátrixok Cholesky felbontását!

a.)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

b.)

$$B = \begin{bmatrix} 8 & 4 & 2 \\ 4 & 6 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

c.)

$$C = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

d.)

$$D = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

2. Oldjuk meg Cholesky felbontással a következő egyenletrendszert!

$$\begin{array}{rcrcrcrcr} 5x_1 & - & x_2 & - & x_3 & - & x_4 & = & -4 \\ -x_1 & + & 10x_2 & - & x_3 & - & x_4 & = & 12 \\ -x_1 & - & x_2 & + & 5x_3 & - & x_4 & = & 8 \\ -x_1 & - & x_2 & - & x_3 & + & 10x_4 & = & 34 \end{array}$$

3. Hozzuk QR alakra a következő mátrixot!

$$A = \begin{bmatrix} 5 & 12 \\ 12 & 0 \end{bmatrix}$$

4. Oldjuk meg Householder felbontással a következő egyenletrendszert!

$$\begin{array}{rcrcrcr} 2.4759\, x_1 & + & 1.6235\, x_2 & + & 4.6231\, x_3 & = & 0.0647 \\ 1.4725\, x_1 & + & 0.9589\, x_2 & - & 1.3253\, x_3 & = & 1.0475 \\ 2.6951\, x_1 & + & 2.8965\, x_2 & - & 1.4794\, x_3 & = & -0.6789 \end{array}$$